Written Assignment 5

MATH2144, Fall 2018

due Wednesday, the 3^{rd} of October

Problem A

Exercise 1. Suppose

$$f(x) = e^{A \cdot x} \cdot (B \cdot \cos(w \cdot x) + C \cdot \sin(w \cdot x)).$$

Calculate $f'(x)$, and simplify it.

Exercise 2. Suppose

$$g(x) = e^{-x/2} \cdot \cos\left(\frac{\sqrt{3}}{2} \cdot x\right).$$

Show that $g''' = g$.

Problem B

One of the earliest models in population dynamics in ecology ran as follows: if the population at year n is $p(n)$, then for some choice of parameters $\lambda, B > 0$,

$$p(n+1) = \lambda \cdot (1 - p(n)/B) \cdot p(n),$$

where λ is what the fertility rate would be under ideal conditions—i.e. no crowding, unlimited resources, and so on—and B is the maximum possible population.[1] Letting q be the fraction p/B, i.e. the fraction that p is of the

[1] The idea is that if the population is small, then there's not much restriction on reproduction, so the fertility rate will be close to λ; however, as the population gets closer to the maximum, the fertility will decrease.

maximum possible population, we have the equivalent model

$$q(n+1) = \lambda \cdot (1 - q(n)) \cdot q(n).$$

Population biologists are concerned with the behavior of $q(n)$ as n gets big—that is, with the behavior of the population after a long time. In particular, they might wonder whether or not $\lim_{n\to\infty} q(n)$ exists, and if so, what it might be. (Review limits at infinity in section 2.7 of the text.)

If we define $f(x) = \lambda \cdot x \cdot (1 - x)$, we can rewrite the model as

$$q(n+1) = f(q(n)).$$

Since $q(n+1) = f(q(n))$ for all $n > 0$, we get

$$\begin{aligned}
q(n) &= f(q(n-1)) \\
&= f(f(q(n-2))) \\
&= \cdots \\
&= f(f(\cdots f(q(0))\cdots)) \\
&= (f^{\circ n})(q(0)),
\end{aligned}$$

where we make the following definition:

Definition 1. For any function g, let $g^{\circ n}$ denote the composition $g \circ \cdots \circ g$ of g after itself n times.

The following exercises are about f (which is called the *logistic map*) and properties of $f^{\circ n}$.

Exercise 3. Let $\lambda = 2$. Graph $y = (f^{\circ n})(x)$ for $n = 0, 1, 2$ and 4.

The graphs suggest that the nonzero point of intersection of $y = x$ and $y = f(x)$ plays an important role. Now, the x-coordinate of this point satisfies the equation $f(x) = x$.

Definition 2. A *fixpoint of a function* g is a point x such that $g(x) = x$.

Exercise 4. Solve for the nonzero fixpoint Q of f in terms of λ.

The graphs also suggest that $(f^{\circ n})(x)$ might approach a constant function away from the endpoints 0 and 1 of [0,1]. Constant functions are functions whose derivatives are 0. So we should look at the derivatives of these iterated compositions $f^{\circ n}$.

Exercise 5. Suppose g is a function such that $g(a) = a$ and $g'(a) = r$.

1. Calculate $(g \circ g)'(a)$, $(g \circ g \circ g)'(a)$, and $(g \circ g \circ g \circ g)'(a)$. Guess a general formula for $(g^{\circ n})'(a)$.

2. Based on your guess, determine

$$\lim_{n \to \infty} |(g^{\circ n})'(a)|$$

   assuming $|r| > 1$.

By the second part, we know that the graphs of $y = (f^{\circ n})(x)$ can't approach constant functions if $|f'(Q)| > 1$.

Exercise 6. Find the smallest B such that $|f'(Q)| > 1$ whenever $\lambda > B$.

Thus, if the population is modelled by $q(n)$ for $\lambda > B$, then the population doesn't get closer to some single value as the years go by.

Exercise 7. Suppose $\lambda = 3.835$. Describe the behavior of $q(n)$ as n gets large, after picking some initial $q(0)$ between 0 and 1.

Bonus 1. Let $\lambda = 2$. (A similar argument with different numbers works for every λ between 1 and B. A more technical argument works for $\lambda = B$.)

1. Show that if $x \in (1/3, 1)$, then

$$\frac{|f(x) - Q|}{|x - Q|} < 1/2.$$

2. Show that if $x \in (1/3, 1)$, then

$$\lim_{n\to\infty} (f^{\circ n})(x) = Q.$$

3. Show that if $x \in (0, 1)$, then

$$\lim_{n\to\infty} (f^{\circ n})(x) = Q.$$