Written Assignment 4

MATH2144, Fall 2018

due Wednesday, the 26^{th} of September

This assignment is largely a test of your algebra and basic calculus skills; it isn't particularly conceptual.

Suppose we have three points $(-1, A)$, $(0, B)$ and $(1, C)$, and two slope choices $\alpha, \gamma \in \mathbf{R}$. We want to define the nicest possible function σ on $[-1, 1]$ such that

- $\sigma(-1) = A$, $\sigma(0) = B$, and $\sigma(1) = C$—i.e., the graph of σ passes through all three points;
- $\sigma'(-1) = \alpha$ and $\sigma'(1) = \gamma$; and
- σ is twice-differentiable—in particular, such that $\sigma''(0)$ exists.

Most folks agree that the nicest possible such function is a *cubic spline*. You will define σ in this assignment.

1. Fix some extra slope $\beta \in \mathbf{R}$.

   (a) Let $R(x) = r_3 \cdot x^3 + r_2 \cdot x^2 + r_1 \cdot x + r_0$, where $r_0 = B$, $r_1 = \beta$, $r_2 = 3 \cdot (C - B) - \gamma - 2 \cdot \beta$, and $r_3 = \beta + \gamma - 2 \cdot (C - B)$.
   Show that $R(0) = B$, $R'(0) = \beta$, $R(1) = C$, and $R'(1) = \gamma$.

   (b) Let $L(x) = \ell_3 \cdot x^3 + \ell_2 \cdot x^2 + \ell_1 \cdot x + \ell_0$, where $\ell_0 = B$, $\ell_1 = \beta$, $\ell_2 = 2 \cdot \beta + \alpha - 3 \cdot (B - A)$, and $\ell_3 = \beta + \alpha - 2 \cdot (B - A)$.
   Show that $L(0) = B$, $L'(0) = \beta$, $L(-1) = A$, and $L'(-1) = \alpha$.

As it turns out, since we used continuous, differentiable pieces to define σ, the fact that $L(0) = B = R(0)$ and $L'(0) = \beta = R'(0)$ implies that σ is continuous and differentiable, even at 0. However, σ might not be twice-differentiable, depending on the choice of β.

2. Find β such that $L''(0) = R''(0)$.

If we pick that β, then σ will be twice-differentiable. Such a σ is called a *cubic spline interpolation*. Such functions are used extensively in CAD and vector graphics software.

Bonus 0) Show that if f, g are continuous at a, and $f(a) = g(a)$, then the function s defined by

$$s(x) = \begin{cases} f(x), & x \leq a \\ g(x), & x > a \end{cases}$$

is continuous at a.

Bonus 1) Show that if f, g are differentiable at a and $f(a) = g(a)$ and $f'(a) = g'(a)$, then the function s defined above is differentiable at a.